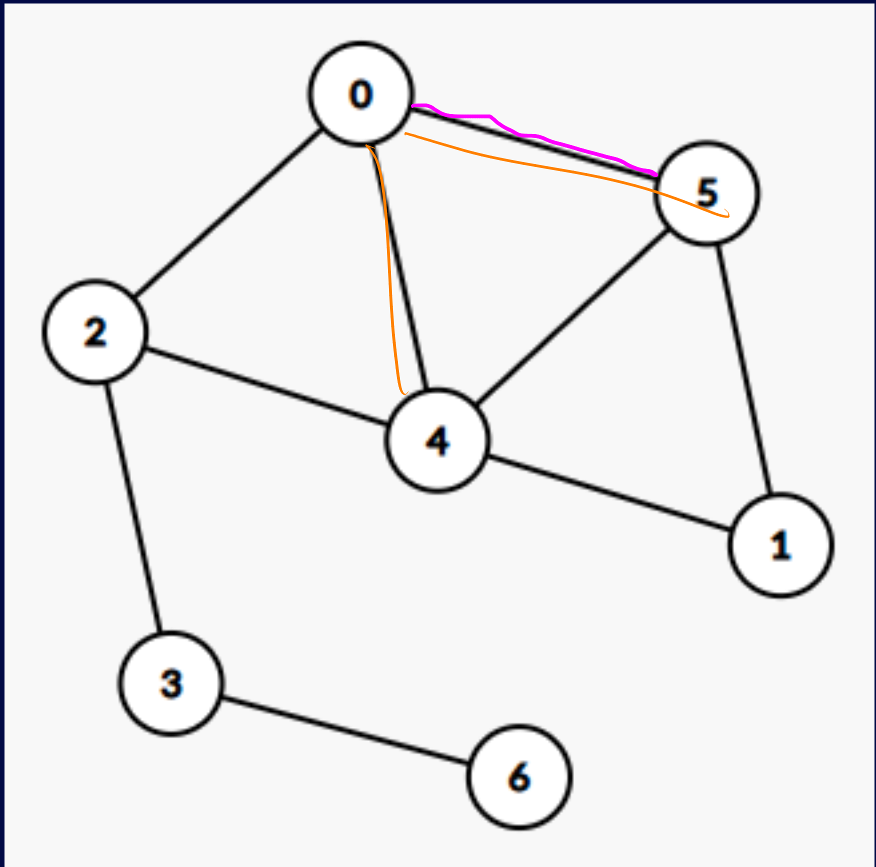
Mathematical Structure and Properties of Graphs



nodes, edges connecting 2 nodes

— Adjacency List/Matrix

$G(V, E)$

graph

vertex set

edge set

$$V = \{v_1, v_2, \dots, v_n\}$$

$$E = \{(v_1, v_3), (v_2, v_3), (v_4, v_5) \dots\}$$

$$E \subseteq V^2$$

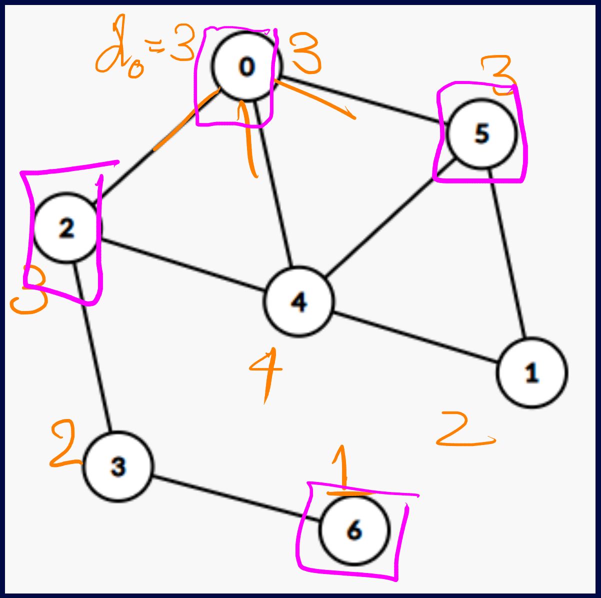
$$V = \{0, 1, 2, 3, 4, 5, 6\}$$

$$E = \{(0,5), (0,4), (4,5), (1,4), \dots\}.$$

Degree Revisited

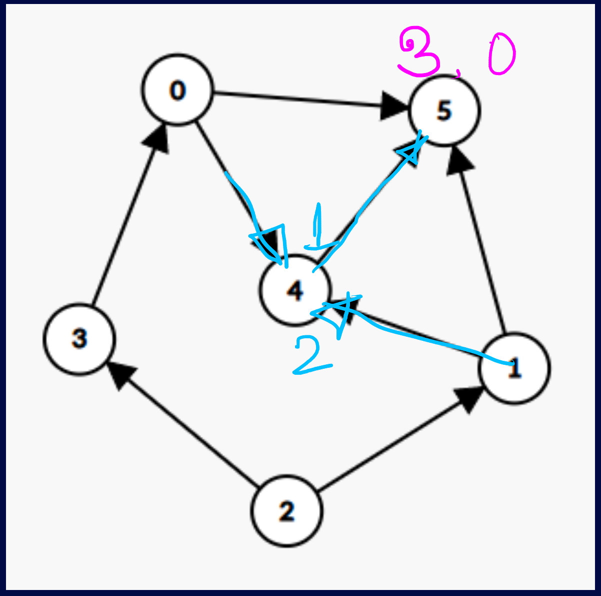
Definition:

degree of u, $d_u \rightarrow$ # of edges connected to u



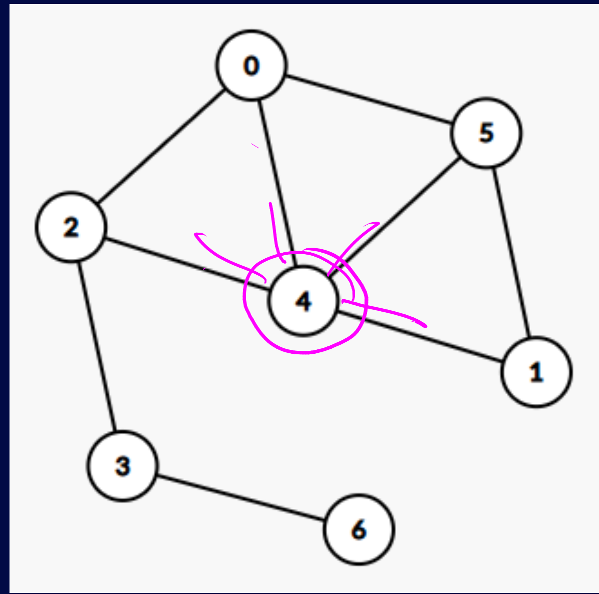
outdegree$_u$, out$_u \rightarrow$ # edges leaving out of u

indegree$_u$, in$_u \rightarrow$ # edges incedent on u



- *Theorem (Handshaking Theorem)*: The sum of degrees of the vertices of an undirected graph is twice the number of edges.

$$\sum_{u \in V} d_u = 2 \cdot |E|$$



out vertices $\longrightarrow$ d_u is odd

even 〃 $\longrightarrow$ d_u is even.

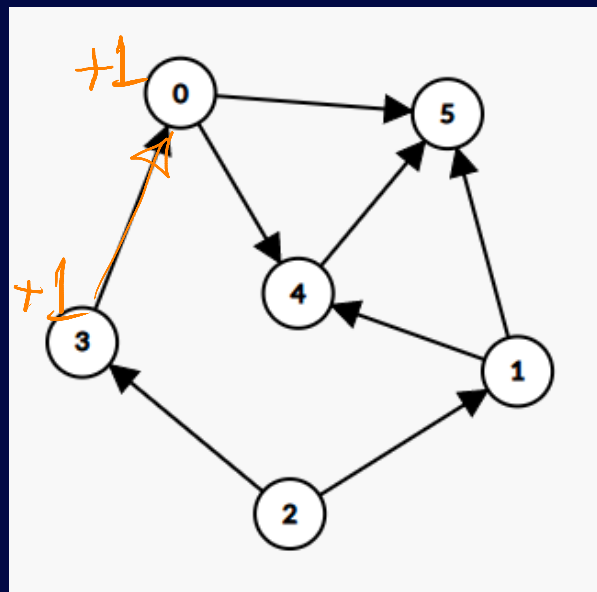
Cor.: $\left|\{u \in V \mid d_u \text{ is odd}\}\right|$ has to be even

- In a directed graph, the sum of outdegrees of all the vertices, the sum of indegrees of all the vertices and the number of edges are equal.

$n, m \le 10^6$

$|E| = m$

$$\left\{ \sum_{u \in V} out_u = \sum_{u \in V} in_u = |E| \right\}$$



DFS $\longrightarrow$ $O\left(n + \sum_{u \in N} d_u\right) = O(n^2)$

$\sum_{u \in N} d_u = 2m = O(m)$

$\longrightarrow$ $O(n+m)$

• *Theorem*: In a group of 6 people, there exists a set of three mutual strangers or three mutual friends.

$S \subseteq \{1, 2, \dots, 6\}$

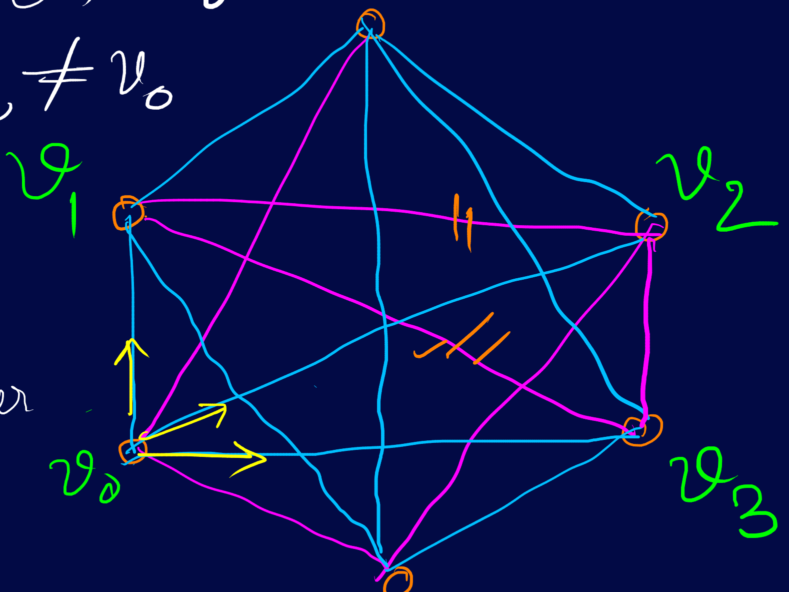
$|S| = 3$.

$S = \{v_1, v_2, v_3\}$

→ choose any vertex v_0

– choose $v_1, v_2, v_3, \neq v_0$ s.t.
$(v_1, v_0), (v_2, v_0), (v_3, v_0)$ are of the same colour

– $(v_1, v_2), (v_2, v_3), (v_3, v_1)$



v_0, v_2, v_3

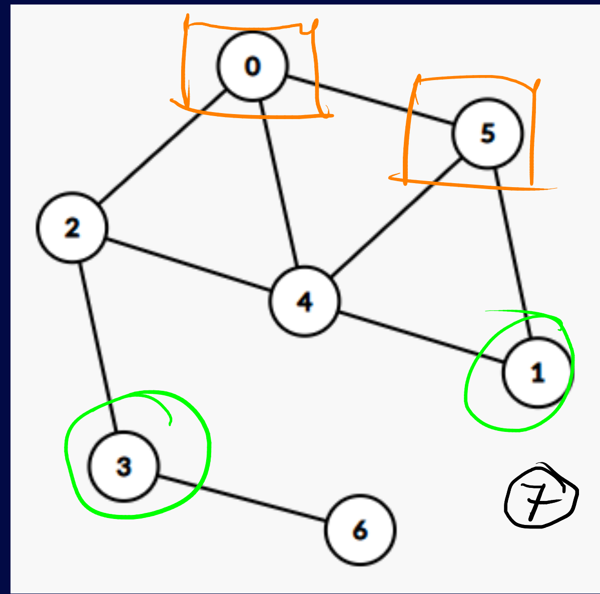
— friends

— strangers

$\{v_1, v_2, v_3\}$

- *Theorem*: In any graph (at least 2 nodes), there are two vertices having the same degree.

$$\exists u, v \in V \text{ such that, } u \neq v \text{ and } d_u = d_v$$



$$d_u \in \underbrace{\{0, 1, 2, \dots, n-1\}}_{n}$$

Assume $\exists\ G(V, E)$ s.t. all degrees are distinct.

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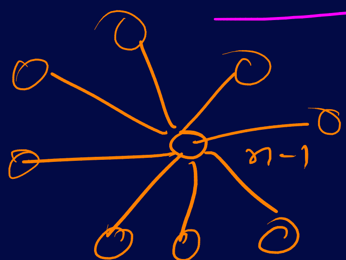
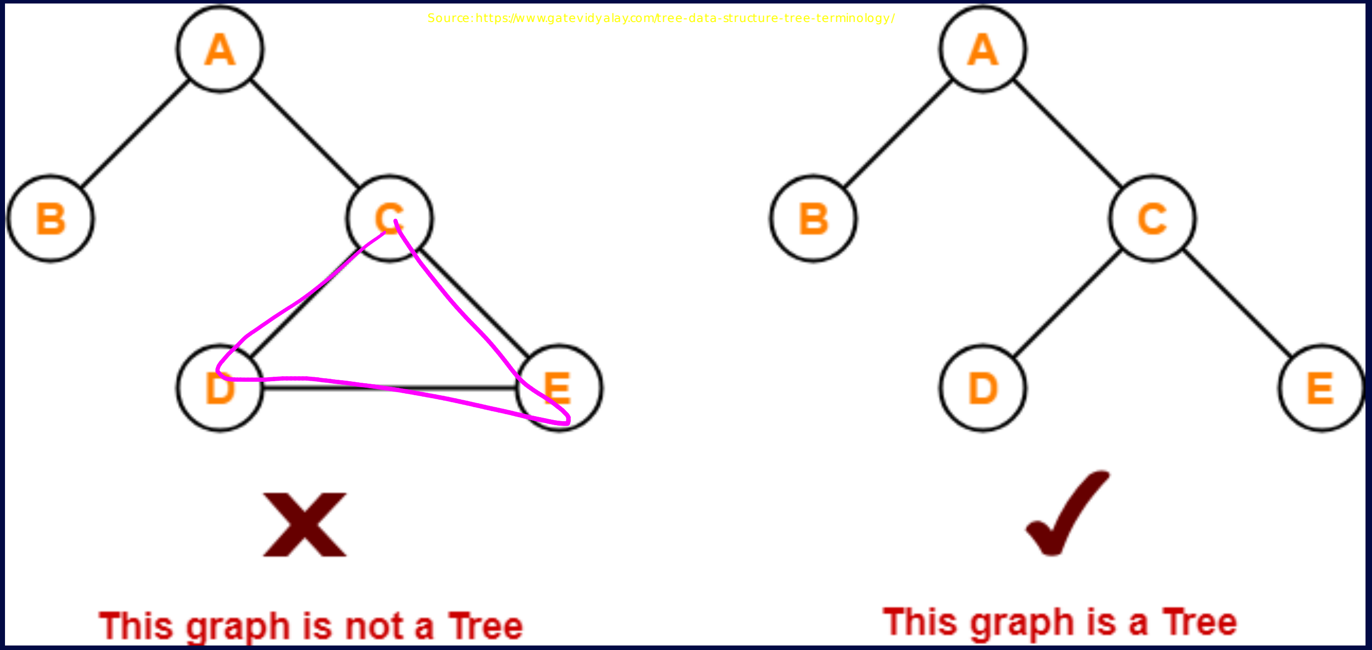
$\{0, 1, 2, \dots 6\}, \{8, \dots, n-1\}$



Trees

Definition: Connected Graphs with no cycles.

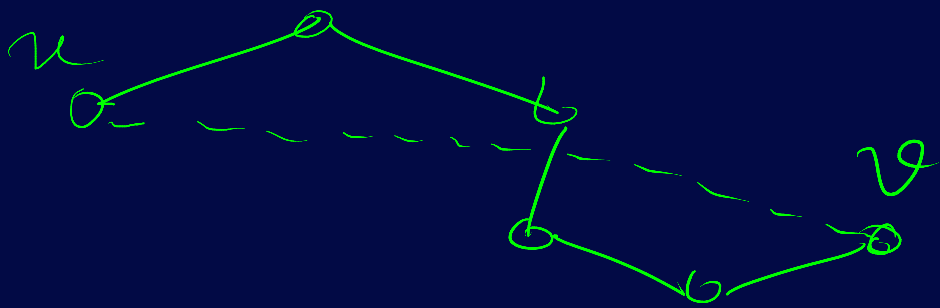


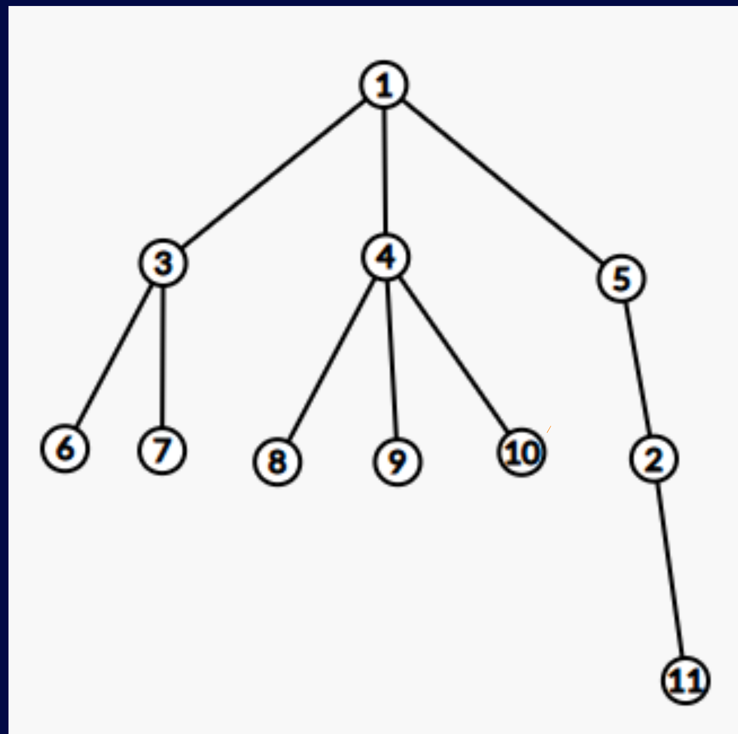
- *Theorem*: The follwing definitions are equivalent:

$$|E| = |V| - 1$$

  - G is a *Tree*
  - G is a connected graph with no cycles
  - G is a connected graph with $|E| = |V| - 1$
  - G is a graph without cycles with $|E| = |V| - 1$
  - Any two vertices in G have exactly one path connecting them
  - G is *Minimally Connected* → if we remove any edge, it becomes disconnected.
  - G is *Maximally Acyclic*

$(u, v) \notin E$ and $u \neq v$





Paths

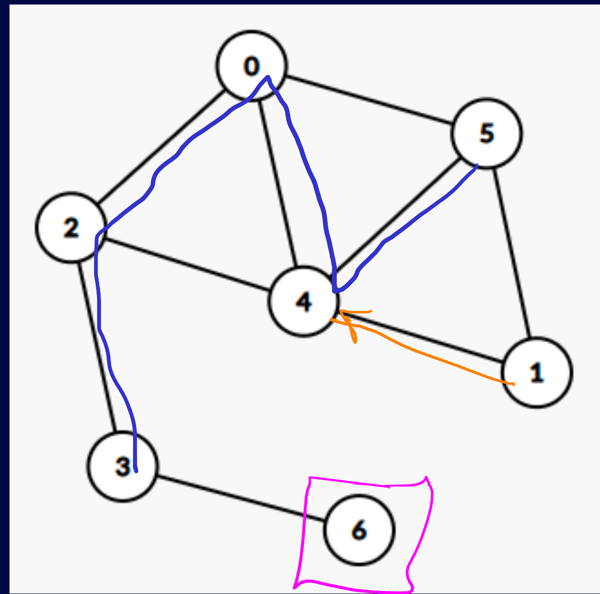
Definition: Seq. of adj. vertices without repetition

(5, 4, 0, 2, 3)

$(5 \to 4) \to 0 \to 2 \to 3$

(1, 4)

(6) $\qquad (5 \to 4)$



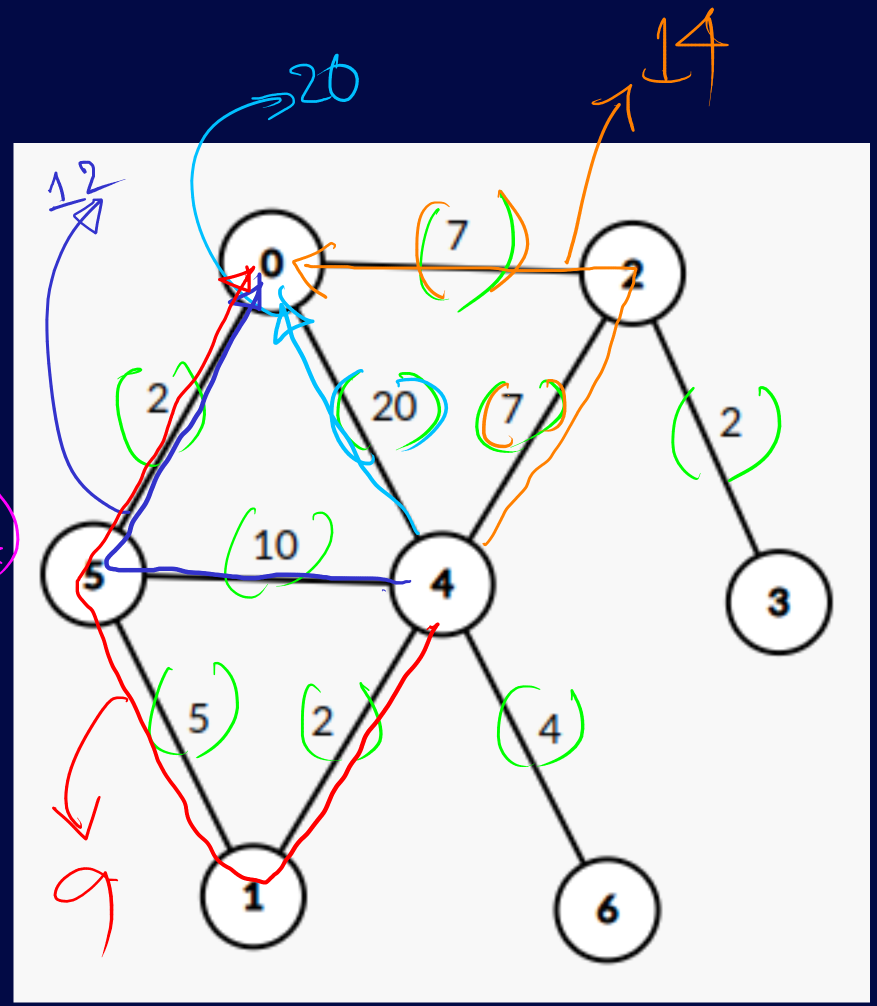
— No edges can ever repeat in a path

Shortest Path Problem:

Length (Cost / Weight) of a path

$v_1 \to v_2 \to v_3 \to \cdots \to v_k$

is $w_{(v_1, v_2)} + w_{(v_2, v_3)} + \cdots + w_{(v_{k-1}, v_k)}$



$d(4, 0) = 9$

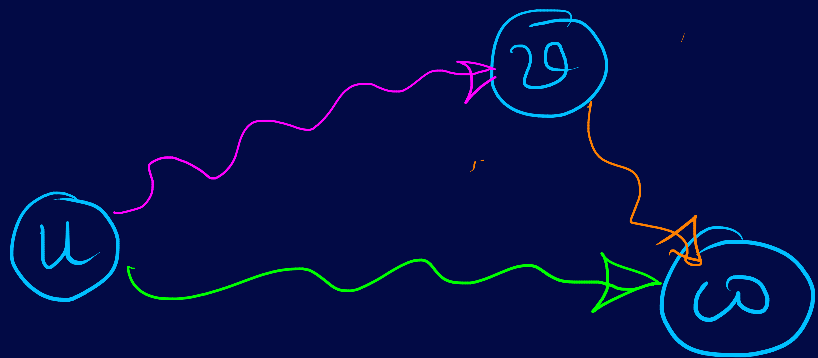
$\mathrm{dist}(u, v)$

$\hookrightarrow$ Length of shortest path from u to v

$\mathrm{dist} : V^2 \to \mathbb{Z}$

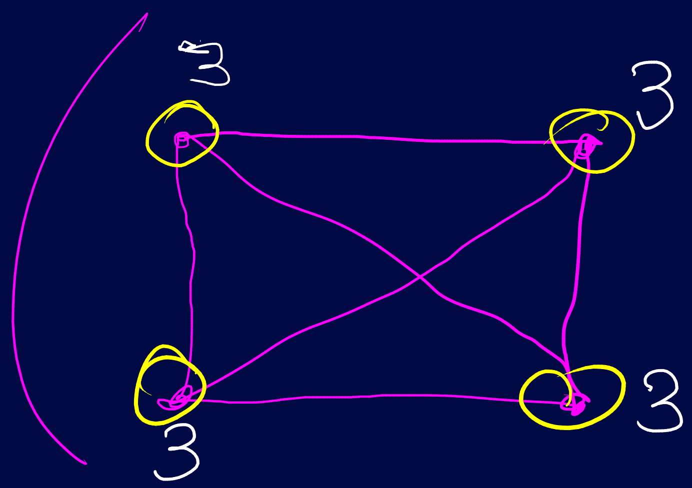
- $\text{dist}(u,u) = 0$
- $\text{dist}(u,v) = \text{dist}(v,u)$ ⟸ Undirected Graphs.
- $\text{dist}(u,v) + \text{dist}(v,w) \geq \text{dist}(u,w)$

$\exists\, u, v, w$ $\quad \underline{\text{dist}(u,v)} + \underline{\text{dist}(v,w)} < \underline{\text{dist}(u,w)}$



Eulerian Trail:

- Something Familiar(?):



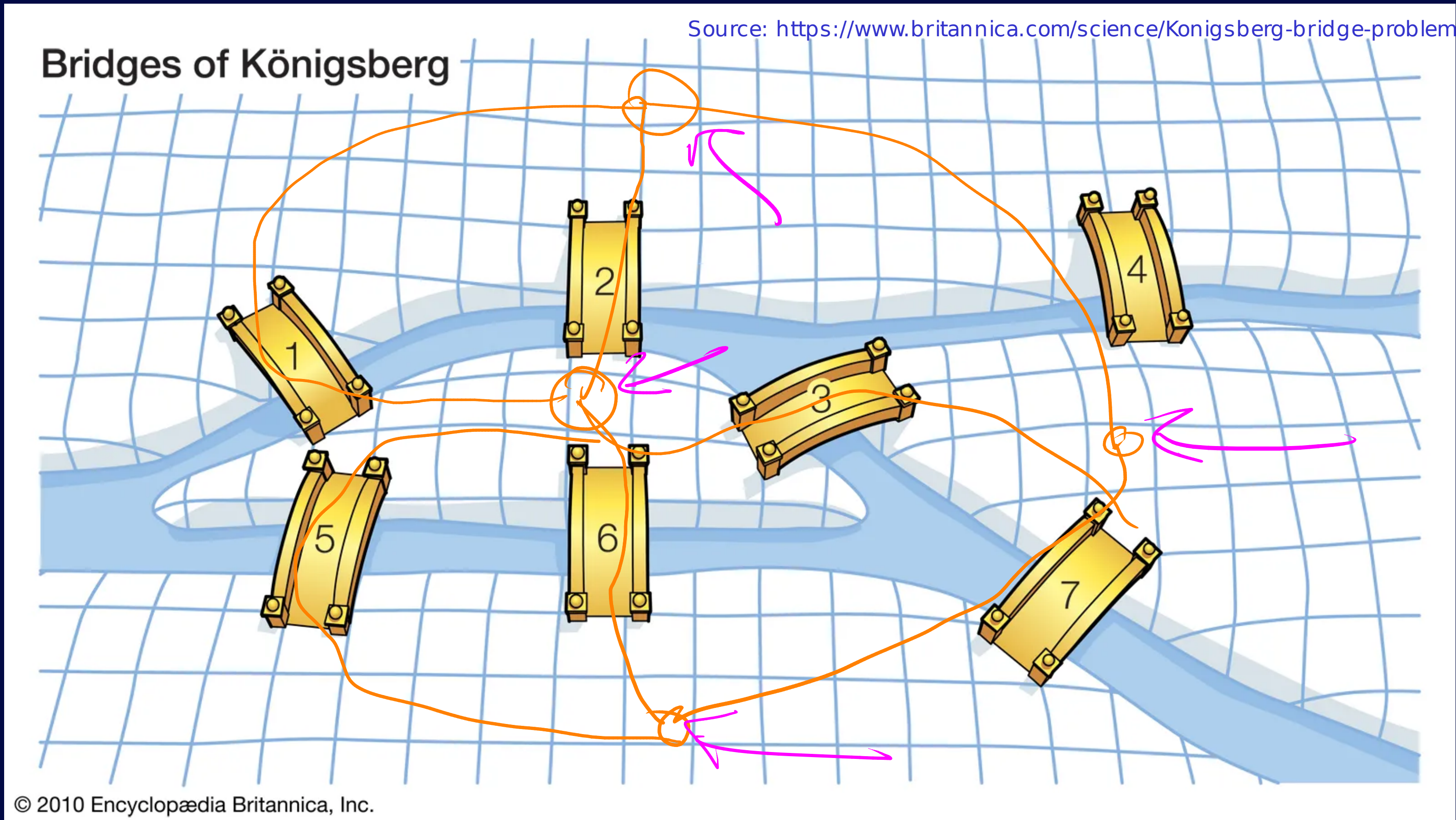
Cannot lift the pen

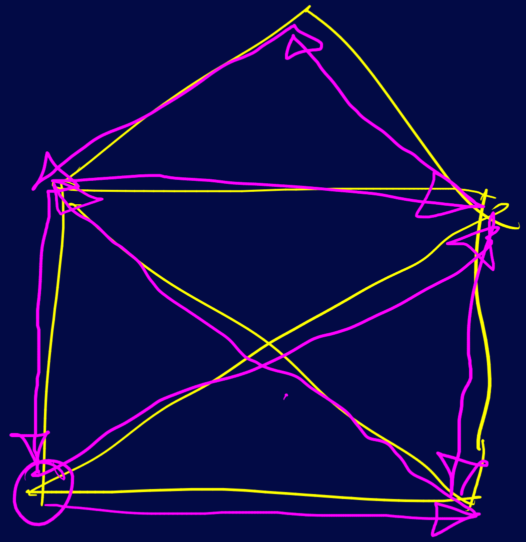
Cannot repeat edges

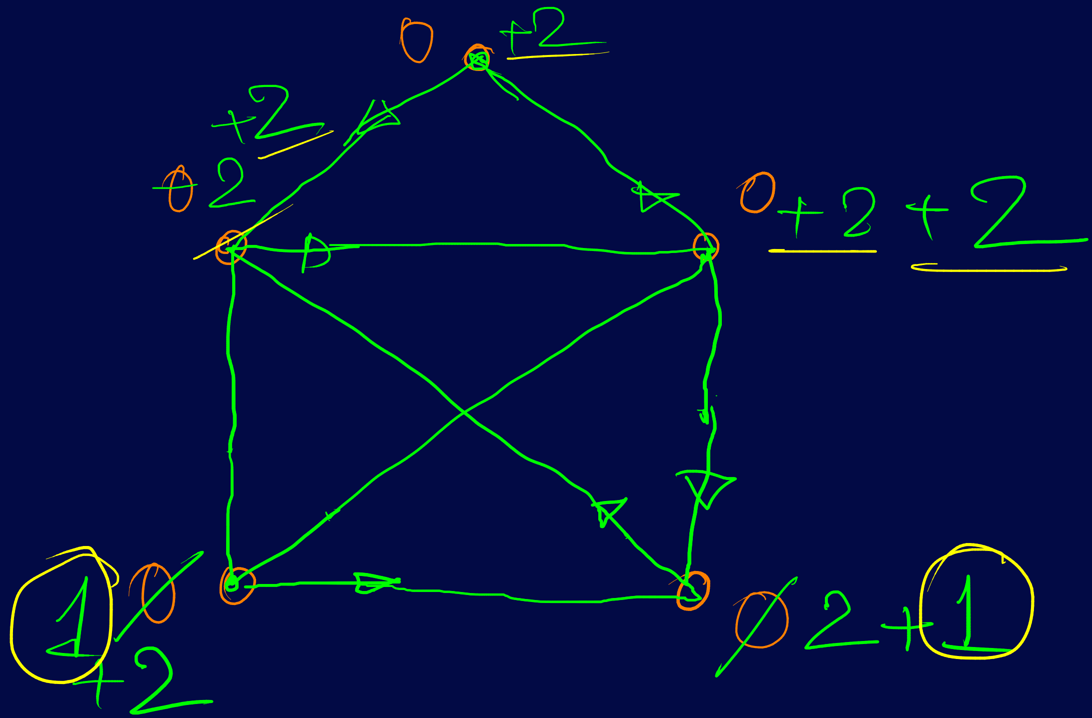
— Find a walk s.t. all edges are covered exactly once.

- Seven Bridges of *Königsberg*

According to folklore, the question arose of whether a citizen could take a walk through the town in such a way that each bridge would be crossed exactly once.






0 +2
+2
0 2
0 + 2 + 2
1 0
+2
0 2+1